LAMBERT'S PROBLEM

Interplanetary Mission Design

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INTRODUCTION

Lambert's problem is a way to solve for the trajectory connecting two position vectors with a given time of flight. In Figure 1, $\vec{r}_0$ and $\vec{r}_f$ define the positions of the initial planet (P_0) at the time of departure and the final planet (P_f) at the time of arrival. With these positions and the time of flight, solving Lambert's problem will define the orbital elements of the transfer orbit. Once the orbital elements of the transfer orbit are computed, the velocities at departure and arrival may be found.

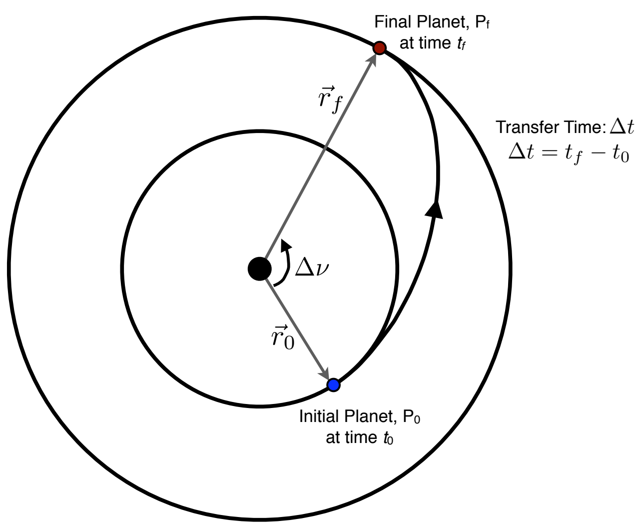

Figure 1. Lambert's Problem

Lambert's Theorem

According to Lambert's Theorem, the transfer time Δt from P_0 to P_f is independent of the orbit's eccentricity and depends only on the sum of the magnitudes of the position vectors, the semimajor axis a and the length of the chord joining P_0 and P_f.

There are many solutions to Lambert's problem that may be found in literature. This handout will detail the Universal Variables Formulation. The definitions of the variables used in this handout may be found in Table 1.

Table 1. Notation

a	=	Semimajor axis
e	=	Eccentricity
E	=	Eccentric Anomaly
i	=	Inclination
k	=	Integer number of revolutions
r	=	Position Vector Magnitude ($r = \vec{r}/\lvert\vec{r}\rvert$)
Δt	=	Time of flight
μ	=	Gravitational Parameter
ν	=	True Anomaly

FORMULATION

Begin with the general form of Kepler's equation:

$$t_f - t_0 = \Delta t = \sqrt{\frac{a^3}{\mu}}\left[2\pi k + (E_f - e\sin E_f) - (E_0 - e\sin E_0)\right], \tag{1}$$

where the subscripts '0' and 'f' refer to the initial and final states, respectively. In this formulation, we are not concerned with multiple revolutions, so we will ignore the $2\pi k$ term. The equation is rearranged to obtain:

$$\begin{aligned} \sqrt{\mu}\Delta t &= \sqrt{a^3}\left[\Delta E + e\left(\sin E_0 - \sin E_f\right)\right] \\ \sqrt{\mu}\Delta t &= \sqrt{a^3}\Delta E + \sqrt{a^3}e\left(\sin E_0 - \sin E_f\right). \end{aligned} \tag{2}$$

We will the use universal variable χ to rearrange Equation 2 into quantities that are already known. Although not provided here, Bate, Mueller, & White[1] and Vallado[2] provide detailed explanations on how to derive the universal variable χ. The expression for χ is

$$\chi = \sqrt{a}\,(E_f - E_0) = \sqrt{a}\,\Delta E. \tag{3}$$

Furthermore, define the constants c_2 and c_3 as

$$c_2 = \frac{1 - \cos\Delta E}{\Delta E^2} \tag{4}$$

$$c_3 = \frac{\Delta E - \sin\Delta E}{\Delta E^3}. \tag{5}$$

The quantity $\chi^3 c_3$ is computed and rearranged:

$$\begin{aligned} \chi^3 c_3 &= \left(\sqrt{a}\Delta E\right)^3 \frac{\Delta E - \sin\Delta E}{\Delta E^3} \\ \chi^3 c_3 &= \sqrt{a^3}\Delta E - \sqrt{a^3}\sin\Delta E \\ \sqrt{a^3}\Delta E &= \chi^3 c_3 + \sqrt{a^3}\sin\Delta E. \end{aligned} \tag{6}$$

Insert Equation 6 into Equation 2 to obtain:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\sin\Delta E + \sqrt{a^3}\,e\left(\sin E_0 - \sin E_f\right). \tag{7}$$

Use the trigonometric identity

$$\sin\Delta E = \sin E_f \cos E_0 - \cos E_f \sin E_0, \tag{8}$$

and Equation 7 becomes

$$\begin{aligned} \sqrt{\mu}\Delta t &= \chi^3 c_3 + \sqrt{a^3}\big(\sin E_f \cos E_0 - \cos E_f \sin E_0 + e\sin E_0 - e\sin E_f\big) \\ \sqrt{\mu}\Delta t &= \chi^3 c_3 + \sqrt{a^3}\big[\sin E_0\,(e - \cos E_f) - \sin E_f\,(e - \cos E_0)\big]. \end{aligned} \tag{9}$$

The sine and cosine functions for true anomaly are given by:

$$\cos\nu = \frac{e - \cos E}{e\cos E - 1} \tag{10}$$

$$\sin\nu = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \tag{11}$$

We wish to express the terms in Equation 9 in terms of the eccentric anomaly *and* the true anomaly. Therefore, we will multiply the final term in Equation 9 by β where:

$$\beta = 1 = \frac{\sqrt{1-e^2}(1-e\cos E_0)(1-e\cos E_f)}{\sqrt{1-e^2}(1-e\cos E_0)(1-e\cos E_f)} \tag{12}$$

Multiplying by β and rearranging gives:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left[\frac{\sin E_0\,(e-\cos E_f) - \sin E_f\,(e-\cos E_0)}{(1-e\cos E_0)(1-e\cos E_f)}\right]\frac{(1-e\cos E_0)(1-e\cos E_f)\sqrt{1-e^2}}{\sqrt{1-e^2}} \tag{13}$$

Now, gather the terms in Equation 13 into terms consistent with Equations 10 and 11:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\left[\frac{\sqrt{1-e^2}\sin E_0}{(1-e\cos E_0)}\frac{e-\cos E_f}{(1-e\cos E_f)} - \frac{\sqrt{1-e^2}\sin E_f}{(1-e\cos E_f)}\frac{e-\cos E_0}{(1-e\cos E_0)}\right]\alpha, \tag{14}$$

where

$$\alpha = \frac{(1-e\cos E_0)(1-e\cos E_f)}{\sqrt{1-e^2}}. \tag{15}$$

Write Equation 14 in terms of $\sin\nu$ and $\cos\nu$:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sqrt{a^3}\big(\sin\nu_0(-\cos\nu_f) - \sin\nu_f(-\cos\nu_0)\big)\left[\frac{(1-e\cos E_0)(1-e\cos E_f)}{\sqrt{1-e^2}}\right]. \tag{16}$$

Use the trig identity given in Equation 8 and the fact that $\sqrt{a^4}/\sqrt{a} = \sqrt{a^3}$ to obtain:

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \sin(\Delta\nu)\frac{a(1-e\cos E_0)a(1-e\cos E_f)}{\sqrt{a(1-e^2)}}. \tag{17}$$

The magnitude of the position vector can be expressed in terms of a, e, and E as follows:

$$r = a(1 - e\cos E). \tag{18}$$

Therefore,

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \frac{r_0 r_f \sin\Delta\nu}{\sqrt{a(1-e^2)}} \frac{\sqrt{1-\cos\Delta\nu}}{\sqrt{1-\cos\Delta\nu}} \tag{19}$$

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + \frac{\sqrt{r_0 r_f} \sin\Delta\nu}{\sqrt{1-\cos\Delta\nu}} \frac{\sqrt{r_0 r_f}\sqrt{1-\cos\Delta\nu}}{\sqrt{a(1-e^2)}}. \tag{20}$$

Let the variables A and y be defined as:

$$A = \frac{\sqrt{r_0 r_f} \sin\Delta\nu}{\sqrt{1-\cos\Delta\nu}} \tag{21}$$

$$y = \frac{r_0 r_f \left(1-\cos\Delta\nu\right)}{a\left(1-e^2\right)}. \tag{22}$$

and Equation 20 may be written as

$$\sqrt{\mu}\Delta t = \chi^3 c_3 + A\sqrt{y}. \tag{23}$$

In terms of the time of flight:

$$\Delta t = \frac{\chi^3 c_3 + A\sqrt{y}}{\sqrt{\mu}}. \tag{24}$$

This is the universal variable form that can be found in Bate, Mueller, & White, and Vallado. Many of the variables can also be expressed equivalently using the variable c_2 given in Equation 4:

$$\chi = \sqrt{\frac{y}{c_2}} \tag{25}$$

$$A = \mathrm{DM}\sqrt{r_0 r_f \left(1+\cos\Delta\nu\right)} \tag{26}$$

$$y = r_0 + r_f + \frac{A\left(\Delta E^2 c_3 - 1\right)}{\sqrt{c_2}}, \tag{27}$$

where

$$\mathrm{DM} = \text{Direction of Motion} = \begin{cases} \text{+1} & \text{if } \Delta\nu < \pi \\ \text{-1} & \text{if } \Delta\nu > \pi. \end{cases}$$

Many texts also replace the quantity ΔE^2 with ψ $(\psi = \Delta E^2)$.

A simple way to derive these equation in terms of ψ is to use the f and g expressions. The derivations for f and g may be found in Bate, Mueller, & White (see Chapter 4). Then, one can iterate on ψ until the desired time of flight is obtained. Vallado uses a secant method. The method is presented in § 7.6, pp. 463 (for the 2^{nd} edition) and is also given in the next section. The solution to Lambert's problem then is an iterative process on ψ, where the variables χ, c_2, c_3, and y are computed for each new value of ψ. The process continues until the desired time of flight is achieved. Note that the value of A is only a function of the magnitude of the two positions and the change in the true anomaly, and is not updated.

ALGORITHM

Input

$\vec{r}_0$	=	Initial Position Vector
$\vec{r}_f$	=	Final Position Vector
Δt_0	=	Desired Transfer Time
DM	=	Direction of Motion** $(-1$ or $+1)$

**An input of DM is *OPTIONAL!* You may pass it in if you want to force the direction of motion to be $(-1$ or $+1)$. Alternatively, you can use the value of $\vec{r}_0$ and $\vec{r}_f$ to compute $\Delta\nu$, which gives you the DM (see hints page).

Initialize the following values

Optional Computation: If DM is not an input

$\Delta\nu = \nu_2 - \nu_1$

** See hints page on computing $\Delta\nu$

IF $\Delta\nu < \pi$,
DM $= 1$
ELSE
DM $= -1$
END IF

$$\cos\Delta\nu = \frac{\vec{r}_0 \bullet \vec{r}_f}{|\vec{r}_0||\vec{r}_f|}$$

$$A = \text{DM}\sqrt{r_0 r_f\,(1 + \cos\Delta\nu)}$$

IF $\Delta\nu = 0$, $A = 0$
Trajectory can't be computed.
END IF

$$c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{6}$$

$$\psi = 0$$

$$\psi_{up} = 4\pi^2$$

$$\psi_{low} = -4\pi$$

WHILE $|\Delta t - \Delta t_0| > 1 \times 10^{-6}$

$$y = r_0 + r_f + \frac{A\,(\psi c_3 - 1)}{\sqrt{c_2}}$$

IF $(A > 0.0 \;\&\; y < 0.0)$

Readjust ψ_{low} until $y > 0.0$ **See Hints Section on ways to adjust ψ

END IF

$$\chi = \sqrt{\frac{y}{c_2}}$$

$$\Delta t = \frac{\chi^3 c_3 + A\sqrt{y}}{\sqrt{\mu}}$$

IF $(\Delta t \leq \Delta t_0)$

$$\psi_{low} = \psi$$

ELSE

$$\psi_{up} = \psi$$

END IF

$$\psi = \frac{\psi_{up} + \psi_{low}}{2}$$

IF $(\psi > 1 \times 10^{-6})$

$$c_2 = \frac{1.0 - \cos\sqrt{\psi}}{\psi}, \qquad c_3 = \frac{\sqrt{\psi} - \sin\sqrt{\psi}}{\sqrt{\psi^3}}$$

ELSE IF $(\psi < -1 \times 10^{-6})$

$$c_2 = \frac{1.0 - \cosh\sqrt{-\psi}}{\psi}, \qquad c_3 = \frac{\sinh\sqrt{-\psi} - \sqrt{-\psi}}{\sqrt{(-\psi)^3}}$$

ELSE

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{6}$$

END IF

END WHILE

Compute

$$f = 1 - \frac{y}{r_0}, \qquad \dot{g} = 1 - \frac{y}{r_f}, \qquad g = A\sqrt{\frac{y}{\mu}}$$

Output

$$\vec{v}_0 = \frac{\vec{r}_f - f\vec{r}_0}{g}, \qquad \vec{v}_f = \frac{\dot{g}\vec{r}_f - \vec{r}_0}{g}$$

Helpful Hints

- You do not necessarily have to pass the value for DM into the code. You may want to pass it in, depending on mission requirements or you may want to compute it with the code. If you are going to be running the code many times within loops, such as for making Pork Chop Plots, it is probably easiest to compute DM inside the code itself. Although the quantity $\cos\Delta\nu$ is computed within the algorithm, do NOT take the arc cosine to get the value of $\Delta\nu$; this will have a sign ambiguity.

  There is a good way to approximate the values for ν_1 and ν_2 which will account for the signs. If we assume that the positions of the planets are in the ecliptic, then we can use the following to compute ν_1 and ν_2:

  $$\tan\nu = \frac{r_y}{r_x} \tag{28}$$

  In MATLAB, there is a function called **atan2** which will guarantee the correct quadrant when you take the arctangent. You can use that function to compute ν_1 and ν_2, and then compute $\Delta\nu$ and DM. Remember to do a check to make sure $0 < \Delta\nu < 2\pi$.
  NOTE: Because we assumed the planets positions are in the ecliptic, if you take the cosine of the value of $\Delta\nu$ that is computed using Equation 28, the value will be slightly different than the quantity $\cos\Delta\nu$ that is computed within the algorithm using the dot-product, since it uses all three components of the planets positions. However, this assumption is generally valid for most of the applications in this class.

- Sometimes, the solver won't converge very quickly, or it won't converge at all for very long or very short transfers, e.g., you might not be able to find a solution for an Earth-Saturn transfer that takes 12 hours. So it's probably a good idea to put some internal checks into your Lambert's solver to prevent it from even attempting a ridiculous transfer. Don't be too concerned if it doesn't work sometimes, but it should always work for *reasonable* Type I and Type II transfers.

- In the algorithm shown above, you'll notice the line

  Readjust ψ_{low} until $y > 0.0$

  This is taken directly from Vallado's algorithm. The algorithm in Bate, Mueller, and White, simply says, "Adjust the trial value of ψ ..." Neither algorithm give a method for exactly *how* to adjust ψ. In the class, people have used different methods to adjust ψ. The reason for adjusting ψ is to make sure that y is positive because we take the square root of it later.

  We have found that simply incrementing ψ by 0.1 seems to work:

  while $y < 0$
      $\psi = \psi + 0.1$
      Resolve for y
  end

  Recall that y is a function of c_2 and c_3, which change as ψ changes. So, within the while loop, you'll need to recompute values for c_2 and c_3.

  There is another method that works more quickly: Since we know that we want to drive $y > 0$, set y equal to 0 in the equation:

  $$y = r_0 + r_f + \frac{A\left(\psi c_3 - 1\right)}{\sqrt{c_2}}$$

and solve for ψ. Then, you can set multiply that value of ψ by some factor N to ensure that $y > 0$. I use a factor $N = 0.8$. You'll still need to use the while loop and recompute values for y, c_2, and c_3.

$$\psi = N \frac{1}{c_3} \left(1 - \frac{\sqrt{c_2}}{A} (r_0 + r_f)\right)$$

As a side note, when you execute the code, there are very few times when you actually have to adjust ψ, so don't stress too much over how you increment ψ.

Also, if you think of a good way to increment ψ, let us know!

REFERENCES

[1] R. R. Bate, D. D. Mueller, and J. E. White, *Fundamentals of Astrodynamics*. Dover Publications, Inc., New York, 1971.

[2] D. A. Vallado, *Fundamentals of Astrodynamics and Applications*. McGraw-Hill Companies, Inc., 1997.